Complex Ginzburg–Landau equation for self–oscillatory systems under amplitude modulated forcing in a 1:1 resonance

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The dynamics of self-oscillatory extended systems, resonantly forced at a frequency close to that of the natural oscillations (1:1 resonance), is shown to be universally described by a complex Ginzburg–Landau equation containing an inhomogeneous term. The case of amplitude modulated forcing is considered, which generalizes previous studies. Application to the two-frequency forcing in a 1:1 resonance is considered as an example.

I. INTRODUCTION

The periodic forcing of spatially extended self-oscillatory systems is a classical method to excite the formation of spatial patterns in such systems. This kind of forcing admits a universal description when the system is operated near the oscillation threshold, and forcing acts on a $n:m$ resonance, defined by the relation $\omega_e = (n/m)(\omega_0 + \nu)$ between the external forcing frequency ω_e and the natural frequency of oscillations ω_0, where n/m is an irreducible integer fraction and ν is a small mistuning. In such case the slowly varying complex amplitude of the oscillations u verifies the following generalized complex Ginzburg-Landau equation (CGLE) [1]

$$\partial_t u = a_1 u + a_2 \nabla^2 u + a_3 |u|^2 u + a_4 \bar{u}^{n-1}, \tag{1}$$

where $\bar{u}$ stands for the complex conjugate of u, $a_{i=1,2,3}$ are complex coefficients and a_4 is proportional to the mth power of the forcing amplitude. Eq. (1) is valid in principle for perfectly periodic forcings and has been introduced making use of symmetry arguments [1]. That elegant reasoning is based on the existence of a discrete time invariance when a perfectly periodic forcing is applied to a system. However if, *e.g.*, the amplitude of forcing is modulated in time in an arbitrary way all temporal symmetries are broken and Eq. (1) is not rigorously justified. In this work we prove that Eq. (1) does provide a universal description of the close to threshold dynamics of self-oscillatory extended systems forced in a 1:1 resonance ($n = 1$ and $m = 1$ in Eq. (1)). The analysis is based on the technique of multiple scales and generalizes the concept of resonant forcing as it considers (almost) periodic forcings that are nonuniform across the system and/or modulated in time. Our derivation hence allows a rigorous simplified study of both noisy forcings as well as multi-frequency forcings within a 1:1 resonance. Generalizations to other resonances ($n \neq 1$ or $m \neq 1$), although cumbersome, can be made straightforwardly following the lines of the present derivation.

II. MODEL

We consider a generic two-dimensional system described by N real dynamical variables $\{U_i(\mathbf{r}, t)\}_{i=1}^{N}$ whose time evolution is governed by the following set of real equations written in vector form,

$$\partial_t \mathbf{U}(\mathbf{r}, t) = \mathbf{f}\left(\mu, \alpha; \mathbf{U}, \nabla^2 \mathbf{U}\right), \tag{2}$$

where $\mathbf{f}$ is a sufficiently differentiable function of its arguments. We assume that the dependence of $\mathbf{f}$ on spatial derivatives of $\mathbf{U}$ is through its Laplacian $\nabla^2 = \partial_x^2 + \partial_y^2$, $\mathbf{r} = (x, y)$. This is the simplest dependence on derivatives in rotationally invariant systems at the time it corresponds to physical systems of most relevance like, *e.g.*, reaction-diffusion and nonlinear optical systems. μ is the bifurcation parameter and $\alpha(\mathbf{r}, t)$ is the forcing parameter, which is allowed to vary on time and space. Physically α may represent either an independent parameter, or the modulated part of any other parameter.

We assume that in the absence of forcing ($\alpha = 0$) Eq. (2) supports a steady, spatially homogeneous state $\mathbf{U} = \mathbf{U}_s(\mu)$ ($\partial_t \mathbf{U}_s = \partial_x \mathbf{U}_s = \partial_y \mathbf{U}_s = 0$), which looses stability at $\mu = \mu_0$ giving rise to a self-oscillatory, spatially homogeneous state. In other words, we assume that the reference state $\mathbf{U}_s$ suffers a homogeneous Hopf bifurcation at $\mu = \mu_0$. We wish to study the small amplitude solutions that form in the system close to the bifurcation when the arbitrary parameter α is modulated in time with a frequency close to that of the free oscillations.

For the sake of convenience we introduce a new vector

$$\mathbf{u}\left(\mathbf{r},t\right)=\mathbf{U}\left(\mathbf{r},t\right)-\mathbf{U}_s, \tag{3}$$

which measures the deviation of the system from the reference state, in terms of which we rewrite Eq. (2) as a Taylor series,

$$\partial_t \mathbf{u} = \mathbf{F}\left(\mu,\alpha\right) + \mathcal{J}\left(\mu,\alpha\right)\cdot\mathbf{u} + \mathcal{D}\left(\mu,\alpha\right)\cdot\nabla^2\mathbf{u}$$
$$+\mathbf{K}\left(\mu,\alpha;\mathbf{u},\mathbf{u}\right) + \mathbf{L}\left(\mu,\alpha;\mathbf{u},\mathbf{u},\mathbf{u}\right) + \text{h.o.t.}, \tag{4}$$

where h.o.t. denotes terms of higher order than 3 in $\mathbf{u}$ or than 1 in $\nabla^2\mathbf{u}$. These h.o.t. have no influence near the bifurcation ($\mu \simeq \mu_0$) since, as we show below, they are $\mathcal{O}\left(|\mu-\mu_0|^2\right)$ or smaller and only terms up to $\mathcal{O}\left(|\mu-\mu_0|^{3/2}\right)$ contribute to the leading order dynamics of the system whenever the bifurcation is supercritical, which is the case we assume.

The different elements of the expansion (4) are defined as

$$\begin{aligned}
\mathbf{F}\left(\mu,\alpha\right) &= \mathbf{f}_s, \\
\mathcal{J}_{ij}\left(\mu,\alpha\right) &= \left[\partial f_i/\partial U_j\right]_s, \mathcal{D}_{ij}\left(\mu,\alpha\right) = \left[\partial f_i/\partial\nabla^2 U_j\right]_s, \\
\mathbf{K}\left(\mu,\alpha;\mathbf{a},\mathbf{b}\right) &= \tfrac{1}{2!}\sum_{i,j=1}^{N}\left[\partial^2\mathbf{f}/\partial U_i\partial U_j\right]_s a_ib_j, \\
\mathbf{L}\left(\mu,\alpha;\mathbf{a},\mathbf{b},\mathbf{c}\right) &= \tfrac{1}{3!}\sum_{i,j,k=1}^{N}\left[\partial^3\mathbf{f}/\partial U_i\partial U_j\partial U_k\right]_s a_ib_jc_k,
\end{aligned}$$

where $\mathbf{a},\mathbf{b},\mathbf{c}$ are arbitrary vectors and the subscript s denotes $\mathbf{U}=\mathbf{U}_s\left(\mu\right)$. Vector $\mathbf{F}\left(\mu,\alpha\right)$ is subjected to the condition

$$\mathbf{F}\left(\mu,0\right)=0, \tag{5}$$

since in the absence of forcing ($\alpha=0$) the reference state $\mathbf{u}=0$ is a steady state of Eq. (4) by hypothesis. $\mathcal{J}$ and $\mathcal{D}$ are matrices, and vector $\mathbf{K}$ ($\mathbf{L}$) is a symmetric and bilinear (symmetric and trilinear) function of its two (three) last arguments.

III. THE HOPF BIFURCATION

In the absence of forcing the stability of the reference state against small perturbations $\delta\mathbf{u}$ is governed by the following equation

$$\partial_t\delta\mathbf{u} = \mathcal{J}\left(\mu,0\right)\cdot\delta\mathbf{u} + \mathcal{D}\left(\mu,0\right)\cdot\nabla^2\delta\mathbf{u}, \tag{6}$$

obtained upon linearizing Eq. (4) for $\alpha=0$ with respect to $\delta\mathbf{u}$. The general solution to Eq. (6) is a superposition of plane waves of the form

$$\delta\mathbf{u}\left(\mathbf{r},t\right) = \sum_j \mathbf{w}_j\exp\left(\Lambda_j t\right)\exp\left(i\mathbf{k}_j\cdot\mathbf{r}\right), \tag{7}$$

with

$$\Lambda_j\mathbf{w}_j = \mathcal{M}\left(\mu,k_j^2\right)\cdot\mathbf{w}_j, \tag{8}$$
$$\mathcal{M}\left(\mu,k^2\right) = \mathcal{J}\left(\mu,0\right) - k^2\mathcal{D}\left(\mu,0\right), \tag{9}$$

($k^2=\mathbf{k}\cdot\mathbf{k}$) hence eigenvalues and eigenvectors of matrix $\mathcal{M}$ depend on $\mathbf{k}$ only through k^2 as $\mathcal{M}$ does. As we are assuming that the reference state looses stability at $\mu=\mu_0$ via a homogeneous Hopf bifurcation in the absence of forcing, matrix $\mathcal{M}\left(\mu,k^2\right)$ must have a pair of complex-conjugate eigenvalues $\{\Lambda_1,\Lambda_2\} = \left\{\lambda\left(\mu,k^2\right),\bar{\lambda}\left(\mu,k^2\right)\right\}$ (the overbar denotes complex conjugation) governing the instability, *i.e.*:

(i) Close to the bifurcation $\operatorname{Re}\Lambda_{i\geq3}<0$ whilst $\operatorname{Re}\lambda$ can become positive for some k's,

(ii) At the bifurcation $\operatorname{Re}\lambda$ is maximum and null at $k=0$ (the perturbation with largest growth rate is spatially homogeneous):

$$\operatorname{Re}\lambda_0=0,\ \left(\partial_k\operatorname{Re}\lambda\right)_0=0,\ \left(\partial_k^2\operatorname{Re}\lambda\right)_0<0. \tag{10}$$

where, here and in the following,

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(iii) The instability is oscillatory, *i.e.*,

$$\operatorname{Im} \lambda_0 = \omega_0 \neq 0. \tag{11}$$

Finally, the preceding properties imply that:

(iv) All eigenvalues of $\mathcal{M}_0 = \mathcal{J}_0$, see Eq. (9), have negative real part but $\left\{\lambda_0, \overline{\lambda}_0\right\} = \{i\omega_0, -i\omega_0\}$.

For the sake of later use we introduce the right and left eigenvectors of $\mathcal{J}_0$ associated with eigenvalues $\{i\omega_0, -i\omega_0\}$,

$$\begin{aligned}
\mathcal{J}_0 \cdot \mathbf{h} &= i\omega_0 \mathbf{h}, \quad \mathcal{J}_0 \cdot \overline{\mathbf{h}} = -i\omega_0 \overline{\mathbf{h}}, \\
\mathbf{h}^\dagger \cdot \mathcal{J}_0 &= i\omega_0 \mathbf{h}^\dagger, \quad \overline{\mathbf{h}^\dagger} \cdot \mathcal{J}_0 = -i\omega_0 \overline{\mathbf{h}^\dagger},
\end{aligned} \tag{12}$$

where the short-hand notation $\mathbf{h} = \mathbf{w}_1\left(\mu = \mu_0, k^2 = 0\right), \overline{\mathbf{h}} = \mathbf{w}_2\left(\mu = \mu_0, k^2 = 0\right)$ has been introduced. These vectors verify the following orthonormality relations:

$$\mathbf{h}^\dagger \cdot \overline{\mathbf{h}} = 0, \mathbf{h}^\dagger \cdot \mathbf{h} = 1.$$

IV. SCALES

We are interested in determining the small amplitude solutions that from in the system close to the Hopf bifurcation, which we define by

$$\mu = \mu_0 + \varepsilon^2 \mu_2, \tag{13}$$

where ε is a smallness parameter $(0 < \varepsilon \ll 1)$. The study is based on the widely used technique of multiple scales. These spatial and temporal scales appear naturally close to the bifurcation and are those on which the asymptotic dynamics of the system evolves. As is well known, in a homogeneous Hopf bifurcation these slow scales are given by

$$T = \varepsilon^2 t, X = \varepsilon x, Y = \varepsilon y. \tag{14}$$

These scales follow from the behaviour of λ close to the bifurcation, Eq. (13), for values of k close to the most unstable mode $k = 0$:

$$\lambda\left(\mu_0 + \varepsilon^2 \mu_2, k^2\right) = \lambda_0 + (\partial_\mu \lambda)_0 \varepsilon^2 \mu_2 + \frac{1}{2}\left(\partial_k^2 \lambda\right)_0 k^2 + \max\left\{\mathcal{O}\left(\varepsilon^4\right), \mathcal{O}\left(k^2\right)\right\}, \tag{15}$$

where the term $(\partial_k \lambda)_0 k$ has not been included since $(\partial_k \lambda)_0 = 0$ as λ is an even function of k. From this equation we obtain, making use of Eq. (10),

$$\operatorname{Re} \lambda\left(\mu_0 + \varepsilon^2 \mu_2, k^2\right) = (\partial_\mu \operatorname{Re} \lambda)_0 \varepsilon^2 \mu_2 - \frac{1}{2}\left|\partial_k^2 \operatorname{Re} \lambda\right|_0 k^2 + \max\left\{\mathcal{O}\left(\varepsilon^4\right), \mathcal{O}\left(k^2\right)\right\},$$

what indicates that the only modes which can experience linear growth verify $k = \mathcal{O}(\varepsilon)$, hence the asymptotic dynamics of the system exhibits spatial variations on a scale $x, y \sim k^{-1} \sim \varepsilon^{-1}$ and the slow spatial scales $(X = \varepsilon x, Y = \varepsilon y)$ follow. Thus, putting $k = \varepsilon k_1$,

$$\operatorname{Re} \lambda\left(\mu_0 + \varepsilon^2 \mu_2, \varepsilon^2 k_1^2\right) = \varepsilon^2\left[(\partial_\mu \operatorname{Re} \lambda)_0 \mu_2 - \frac{1}{2}\left|\partial_k^2 \operatorname{Re} \lambda\right|_0 k_1^2\right] + \mathcal{O}\left(\varepsilon^4\right),$$

which shows that the growth of perturbations occurs on a scale $t \sim (\operatorname{Re} \lambda)^{-1} \sim \varepsilon^{-2}$ and the slow timescale $T = \varepsilon^2 t$ follows. On the other hand, making use of Eq. (11) and putting $k = \varepsilon k_1$ again, Eq. (15) yields

$$\operatorname{Im} \lambda\left(\mu_0 + \varepsilon^2 \mu_2, \varepsilon^2 k_1^2\right) = \omega_0 + \varepsilon^2\left[(\partial_\mu \operatorname{Im} \lambda)_0 \mu_2 + \frac{1}{2}\left(\partial_k^2 \operatorname{Im} \lambda\right)_0 k_1^2\right] + \mathcal{O}\left(\varepsilon^4\right),$$

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whose first term, $\omega_0 = \mathcal{O}\left(\varepsilon^0\right)$, indicates that the original timescale t must be retained, whilst the rest of terms do not introduce other relevant timescales.

As for the external forcing we assume that its form is consistent with the previous scales. In particular we assume that it is weak and of the form

$$\alpha\left(\mathbf{r}, t\right) = \varepsilon^3 \alpha_3\left(X, Y, T, t\right), \tag{16}$$

$$\alpha_3\left(X, Y, T, t\right) = \sum_{p=1}^{\infty} \left[\alpha_{3,p}\left(X, Y, T\right) \exp\left(ip\omega_0 t\right) + \bar{\alpha}_{3,p}\left(X, Y, T\right) \exp\left(-ip\omega_0 t\right)\right], \tag{17}$$

Thus we are considering a resonant forcing with slowly varying amplitude. Notice that, according to Eqs. (16,17), $\alpha\left(\mathbf{r}, t + \frac{2\pi}{\omega_0}\right) = \alpha\left(\mathbf{r}, t\right) + \mathcal{O}\left(\varepsilon^5\right)$ and hence the considered forcing is almost periodic of fundamental frequency ω_0.

Under these conditions a multiple scale analysis is possible and we look for asymptotic solutions to Eq. (4) in the form

$$\mathbf{u}\left(\mathbf{r}, t\right) = \sum_{m=1}^{\infty} \varepsilon^m \mathbf{u}_m\left(X, Y, T\right). \tag{18}$$

We finally introduce Eqs. (13) and (16–18) into Eq. (4) making use of the following chain rules for differentiation

$$\partial_t \mathbf{u} = \sum_{m=1}^{\infty} \varepsilon^m \left(\partial_t \mathbf{u}_m + \varepsilon^2 \partial_T \mathbf{u}_m\right),$$

$$\nabla^2 \mathbf{u} = \sum_{m=1}^{\infty} \varepsilon^{m+2} \left(\partial_X^2 + \partial_Y^2\right) \mathbf{u}_m,$$

and solve at increasing orders in ε.

V. THE COMPLEX GINZBURG–LANDAU EQUATION

The general form of Eq. (4) at any order ε^m is found to be

$$\mathfrak{J}\left(\mathbf{u}_m\right) = \mathbf{g}_m\left(X, Y, T, t\right), \tag{19}$$

where

$$\mathfrak{J}\left(\mathbf{u}\right) \equiv \partial_t \mathbf{u} - \mathcal{J}_0 \cdot \mathbf{u}, \tag{20}$$

and $\mathbf{g}_m$ does not depend on $\mathbf{u}_m$ (but on $\mathbf{u}_{n<m}$). Clearly, as $\mathfrak{J}\left(\exp\left(i\omega_0 t\right) \mathbf{h}\right) = \mathfrak{J}\left(\exp\left(-i\omega_0 t\right) \overline{\mathbf{h}}\right) = 0$, see Eq. (12), the solvability of Eq. (19) requires

$$\int_t^{t+2\pi/\omega_0} dt' \mathbf{h}^\dagger \cdot \mathbf{g}_m\left(X, Y, T, t'\right) \exp\left(-i\omega_0 t'\right) = 0, \tag{21}$$

(or its equivalent complex-conjugate) which ensures that $\mathbf{g}_m$ does not contain secular terms (proportional to $\exp\left(i\omega_0 t\right) \mathbf{h}$ or to $\exp\left(-i\omega_0 t\right) \overline{\mathbf{h}}$). Once condition (21) is verified, the asymptotic solution to Eq. (19) reads

$$\begin{aligned}\mathbf{u}_m\left(X, Y, T, t\right) &= u_m\left(X, Y, T\right) \exp\left(i\omega_0 t\right) \mathbf{h} + \overline{u}_m\left(X, Y, T\right) \exp\left(-i\omega_0 t\right) \overline{\mathbf{h}} \\ &\quad + \mathbf{u}_m^\perp\left(X, Y, T, t\right),\end{aligned} \tag{22}$$

where $u_m\left(X, Y, T\right)$ is a function of the slow scales, and the last term is the particular solution. Note that the solution (22) should involve, in principle, terms proportional to all the eigenvectors of $\mathfrak{J}\left(\cdot\right)$ [which are those of $\mathcal{J}_0$, see Eq. (8)]. However all of them are damped according to $\exp\left[-\left|\operatorname{Re}\Lambda_i\left(\mu_0, 0\right)\right| t\right]$, since $\operatorname{Re}\Lambda_{i\geq 3}\left(\mu = \mu_0, k = 0\right) < 0$ by hypothesis, except those associated with $\left(\mathbf{h}, \overline{\mathbf{h}}\right)$.

A. Order ε

This is the first nontrivial order and reads

$$\mathbf{g}_1 = 0. \tag{23}$$

The solvability condition (21) at this order is fulfilled and, according to Eq. (22),

$$\mathbf{u}_1 = u_1\left(X, Y, T\right) \exp\left(i\omega_0 t\right) \mathbf{h} + \overline{u}_1\left(X, Y, T\right) \exp\left(-i\omega_0 t\right) \overline{\mathbf{h}}. \tag{24}$$

B. Order ε^2

At this order

$$\mathbf{g}_2 = (\partial_\mu \mathbf{F})_0 \, \mu_2 + \mathbf{K} \left(\mu_0, 0; \mathbf{u}_1, \mathbf{u}_1\right). \tag{25}$$

The first term of the r.h.s. is null by virtue of Eq. (5). Making use of Eq. (24) and taking into account that $\mathbf{K}$ is symmetric and bilinear in its two last arguments, Eq. (25) can be written as

$$\mathbf{g}_2 = 2\mathbf{K} \left(\mu_0, 0; \mathbf{h}, \overline{\mathbf{h}}\right) |u_1|^2 + \mathbf{K} \left(\mu_0, 0; \mathbf{h}, \mathbf{h}\right) u_1^2 \exp\left(i2\omega_0 t\right)$$
$$+ \mathbf{K} \left(\mu_0, 0; \overline{\mathbf{h}}, \overline{\mathbf{h}}\right) \overline{u}_1^2 \exp\left(-i2\omega_0 t\right). \tag{26}$$

The solvability condition (21) is automatically fulfilled again and, according to Eq. (22),

$$\mathbf{u}_2 = u_2 \left(X, Y, T\right) \exp\left(i\omega_0 t\right) \mathbf{h} + \overline{u}_2 \left(X, Y, T\right) \exp\left(-i\omega_0 t\right) \overline{\mathbf{h}}$$
$$+ \mathbf{v}_0 |u_1|^2 + \mathbf{v}_2 u_1^2 \exp\left(2i\omega_0 t\right) + \overline{\mathbf{v}}_2 \overline{u}_1^2 \exp\left(-2i\omega_0 t\right), \tag{27}$$

where,

$$\mathbf{v}_0 = -2\mathcal{J}_0^{-1} \cdot \mathbf{K} \left(\mu_0, 0; \mathbf{h}, \overline{\mathbf{h}}\right), \tag{28}$$

$$\mathbf{v}_2 = -\left(\mathcal{J}_0 - i2\omega_0 \mathcal{I}\right)^{-1} \cdot \mathbf{K} \left(\mu_0, 0; \mathbf{h}, \mathbf{h}\right), \tag{29}$$

are constant vectors, and $\mathcal{I}$ is the $N \times N$ identity matrix. Note that both $\mathcal{J}_0$ and $\left(\mathcal{J}_0 - i2\omega_0 \mathcal{I}\right)$ are invertible since neither 0 nor $2i\omega_0$ are eigenvalues of $\mathcal{J}_0$ by hypothesis: otherwise other eigenvalues different from $\{i\omega_0, -i\omega_0\}$ would have null real part at the bifurcation.

C. Order ε^3

Finally, at this order we find

$$\mathbf{g}_3 = -\partial_T \mathbf{u}_1 + \mu_2 \left(\partial_\mu \mathcal{J}\right)_0 \cdot \mathbf{u}_1 + \mathcal{D}_0 \cdot \left(\partial_X^2 + \partial_Y^2\right) \mathbf{u}_1$$
$$+ \left(\partial_\alpha \mathbf{F}\right)_0 \alpha_3 + 2\mathbf{K} \left(\mu_0, 0; \mathbf{u}_1, \mathbf{u}_2\right) + \mathbf{L} \left(\mu_0, 0; \mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_1\right). \tag{30}$$

Application of the solvability condition (21) yields, after substituting Eqs. (17), (24) and (27) into Eq. (30), and making use of the symmetry and linearity properties of vectors $\mathbf{K}$ and $\mathbf{L}$,

$$\partial_T u_1 = c_1 \mu_2 u_1 + c_2 \left(\partial_X^2 + \partial_Y^2\right) u_1 + c_3 |u_1|^2 u_1 + c_4 \alpha_{3,1}, \tag{31}$$

where

$$c_1 = \mathbf{h}^\dagger \cdot \left(\partial_\mu \mathcal{J}\right)_0 \cdot \mathbf{h}, \tag{32}$$

$$c_2 = \mathbf{h}^\dagger \cdot \mathcal{D}_0 \cdot \mathbf{h}, \tag{33}$$

$$c_3 = 2\mathbf{h}^\dagger \cdot \mathbf{K} \left(\mu_0, 0; \mathbf{h}, \mathbf{v}_0\right) + 2\mathbf{h}^\dagger \cdot \mathbf{K} \left(\mu_0, 0; \overline{\mathbf{h}}, \mathbf{v}_2\right) + 3\mathbf{h}^\dagger \cdot \mathbf{L} \left(\mu_0, 0; \mathbf{h}, \mathbf{h}, \overline{\mathbf{h}}\right), \tag{34}$$

$$c_4 = \mathbf{h}^\dagger \cdot \left(\partial_\alpha \mathbf{F}\right)_0, \tag{35}$$

are constant coefficients.

Finally note that, making use of Eqs. (18), (14) and (24), the asymptotic state of the system can be written as

$$\mathbf{u}\left(\mathbf{r}, t\right) = u\left(\mathbf{r}, t\right) \exp\left(i\omega_0 t\right) \mathbf{h} + \overline{u}\left(\mathbf{r}, t\right) \exp\left(-i\omega_0 t\right) \overline{\mathbf{h}} + \mathcal{O}\left(\varepsilon^2\right), \tag{36}$$
$$u\left(\mathbf{r}, t\right) = \varepsilon u_1\left(X, Y, T, t\right),$$

hence u denotes the leading order amplitude of oscillations. Its evolution equation is obtained from Eq. (31) by returning to original scales and parameters via Eqs. (13) and (14), and reads

$$\partial_t u\left(\mathbf{r}, t\right) = \left(\mu - \mu_0\right) c_1 u + c_2 \nabla^2 u + c_3 |u|^2 u + f\left(\mathbf{r}, t\right), \tag{37}$$

where

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$$f\left(\mathbf{r}, t\right) = c_4 \varepsilon^3 \alpha_{3,1}\left(X, Y, T\right) \tag{38}$$

is a slowly varying function proportional to the complex amplitude of the fundamental component of forcing $\varepsilon^3 \alpha_{3,1}$, see Eqs. (16,17).

Eq. (37), or (31), is a complex Ginzburg–Landau equation containing an inhomogeneous forcing term, $f\left(\mathbf{r}, t\right)$, which generalizes Eq. (1) for $n = m = 1$. Eq. (37) is valid whenever $\operatorname{Re} c_3 \leq 0$ (supercritical bifurcation) since otherwise it could lead to unbounded solutions. If $\operatorname{Re} c_3 > 0$ the bifurcation is subcritical and the analysis must incorporate higher orders in the ε–expansion.

VI. APPLICATION TO TWO-FREQUENCY FORCING

Just for the sake of illustration let us finally consider the two-frequency forcing case. In particular we assume that the forcing parameter α has the form

$$\alpha\left(\mathbf{r}, t\right) = A\left(\omega_1 t\right) + A\left(\omega_2 t\right), \tag{39}$$

where A is a 2π periodic function $[A\left(\theta\right) = A\left(\theta + 2\pi\right)]$. This means that we are dealing with the superposition of two forcings of equal amplitude but of different frequencies (ω_1 and ω_2). An example of this type of forcing could be the illumination of a photosensitive version of the Belousov-Zhabotinsky reaction with two equal sources, whose light intensities are periodically modulated in time at two different frequencies. Another example could be the injection into a laser cavity of two coherent fields of equal amplitudes and different frequencies.

Due to the commented periodicity one can write

$$A\left(\omega_i t\right) = \sum_{p=1}^{\infty} \left[A_p \exp\left(ip\omega_i t\right) + \bar{A}_p \exp\left(-ip\omega_i t\right)\right]. \tag{40}$$

(Note that the term $p = 0$ is absent, as in Eqs. (16,17), since it would correspond to a constant bias. This bias term, if any, is implicitly considered in our analysis at the same level as any other constant parameter of the system and it would appear, in principle, in the determination of the Hopf bifurcation). Upon rewriting Eq. (40) as

$$A\left(\omega_i t\right) = \sum_{p=1}^{\infty} \left[A_p \exp\left(ip\delta_i t\right) \exp\left(ip\omega_0 t\right) + \bar{A}_p \exp\left(-ip\delta_i t\right) \exp\left(-ip\omega_0 t\right)\right], \tag{41}$$

where $\delta_i = \omega_i - \omega_0$, we can express Eq. (39) in the form (16,17):

$$\alpha\left(\mathbf{r}, t\right) = \sum_{p=1}^{\infty} \left[a_p\left(t\right) \exp\left(ip\omega_0 t\right) + \bar{a}_p\left(t\right) \exp\left(-ip\omega_0 t\right)\right], \tag{42}$$

where

$$a_p\left(t\right) = A_p\left[\exp\left(ip\delta_1 t\right) + \exp\left(ip\delta_2 t\right)\right] = 2A_p \exp\left(ip\nu t\right) \cos\left(p\omega t\right), \tag{43}$$

where

$$\nu \equiv \frac{\delta_1 + \delta_2}{2} = \frac{\omega_1 + \omega_2}{2} - \omega_0, \quad \omega \equiv \frac{\delta_1 - \delta_2}{2} = \frac{\omega_1 - \omega_2}{2}. \tag{44}$$

(We note incidentally that, in the case of single-frequency forcing, $\omega_1 = \omega_2 \equiv \omega_e$ and hence $\nu = \omega_e - \omega_0$ ($\omega_e = \omega_0 + \nu$), and $\omega = 0$.) Hence all our previous analysis is valid for this type of driving, whenever forcing is weak ($A_p \sim \varepsilon^3$) and the time scale along which the a_p's vary is slow. Specifically this requires $\nu, \omega \sim \varepsilon^2$. If these relations are satisfied, the system will be described by Eq. (37) with f given by Eq. (38) and $\varepsilon^3 \alpha_{3,1}$ given by $a_{p=1}\left(t\right)$:

$$\partial_t u\left(\mathbf{r}, t\right) = \left(\mu - \mu_0\right) c_1 u + c_2 \nabla^2 u + c_3 \left|u\right|^2 u + 2c_4 A_1 \exp\left(i\nu t\right) \cos\left(\omega t\right). \tag{45}$$

In order to simplify this equation let us define

$$U\left(\mathbf{r}, t\right) = u\left(\mathbf{r}, t\right) \exp\left(-i\nu t\right), \tag{46}$$

which, substituted into Eq. (45) yields

$$\partial_t U\left(\mathbf{r}, t\right) = \left[\left(\mu - \mu_0\right) c_1 - i\nu\right] U + c_2 \nabla^2 U + c_3 \left|U\right|^2 U + 2c_4 A_1 \cos\left(\omega t\right). \tag{47}$$

The actual state of the system, Eq. (36), is given in terms of U as

$$\mathbf{u}\left(\mathbf{r}, t\right) = 2\operatorname{Re}\left\{U\left(\mathbf{r}, t\right)\exp\left[i\left(\omega_0 + \nu\right)t\right]\mathbf{h}\right\}. \tag{48}$$

Finally, note that, especially in nonlinear optics, one is used to express the state of the system in terms of the negative-frequency part of the oscillations, i.e., as

$$\mathbf{u}\left(\mathbf{r}, t\right) = 2\operatorname{Re}\left\{\bar{U}\left(\mathbf{r}, t\right)\exp\left[-i\left(\omega_0 + \nu\right)t\right]\overline{\mathbf{h}}\right\}, \tag{49}$$

and the complex amplitude $\bar{U}\left(\mathbf{r}, t\right)$ verifies the complex conjugate of Eq. (47):

$$\partial_t\bar{U}\left(\mathbf{r}, t\right) = \left[\left(\mu - \mu_0\right)\bar{c}_1 + i\nu\right]\bar{U} + \bar{c}_2\nabla^2\bar{U} + \bar{c}_3\left|\bar{U}\right|^2\bar{U} + 2\bar{c}_4\bar{A}_1\cos\left(\omega t\right). \tag{50}$$

This absolutely trivial transformation is done here just to point out that the sign that affects the mistuning ν in the forced CGL equation depends on whether one uses the amplitude of the positive- or the negative-frequency part of the oscillations in order to describe the dynamics of the system.

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[1] P. Coullet and K. Emilsson, Physica D **61**, 119 (1992); Physica A **188**, 190 (1992).